WAMC10 Solutions

Ryan Tang, Wesley Wu, and more

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The following document contains an answer key and solutions to all 25 problems of the WAMC 10.

1 Answer Key

1. (C) $\frac{5}{24}$
2. (D) $\frac{15}{4}$
3. (C) 115
4. (D) 500
5. (C) 6073
6. (A) 4
7. (E) 217
8. (D) 16260
9. (E) 21
10. (B) 102
11. (B) 8
12. (B) 1620
13. (E) 19
14. (B) 432
15. (A) 4
16. (A) 1184
17. (E) 6700
18. (C) 5426
19. (C) $\frac{1}{48}$
20. (B) 336
21. (A) -71424
22. (E) 33
23. (D) 6
24. (B) 92
25. (A) 26913

2 Solutions

1. What is $\dfrac{1}{\frac{4}{\left(\frac{3}{4}\right)}} + \dfrac{\frac{\left(\frac{1}{4}\right)}{3}}{4}$?
(A) $\frac{1}{24}$ (B) $\frac{1}{12}$ (C) $\frac{5}{24}$ (D) $\frac{17}{48}$ (E) $\frac{2}{3}$

Ryan Tang

Solution: $\dfrac{1}{\frac{4}{\left(\frac{3}{4}\right)}} + \dfrac{\frac{\left(\frac{1}{4}\right)}{3}}{4} = \frac{1}{\frac{16}{3}} + \frac{1}{48} = \frac{3}{16} + \frac{1}{48} = \boxed{\text{(C) } \frac{5}{24}}$.

2. Ryan has a number. He multiplies it by 5, which increases the number by 1. He then multiplies it by 3. What is the number now?
(A) $\frac{1}{4}$ (B) $\frac{3}{4}$ (C) $\frac{5}{4}$ (D) $\frac{15}{4}$ (E) $\frac{25}{4}$

Ryan Tang

Solution: Let x be Ryan's number at the start. $5x = x + 1$, so $4x = 1$ and $x = \frac{1}{4}$. After multiplying by 5, the number becomes $5x = \frac{5}{4}$. Ryan then multiplies the number by 3, which makes it $\boxed{\text{(D) } \frac{15}{4}}$.

3. Planes are supposed to descend to the runway at 141 knots. One flight descends to the runway at 241 knots. One knot is equal to 1.15 miles per hour. How much faster than normal, in miles per hour, is that flight descending?
(A) 87 (B) 100 (C) 115 (D) 120 (E) 126

Wesley Wu

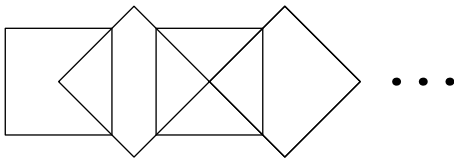
Solution: The difference between the landing speeds is 100 knots. Since 1 knot is equal to 1.15 miles per hour, our answer is $100 \cdot 1.15 = \boxed{\text{(C) } 115}$.

4. Spirit Airlines charges \$25 for a flight from Seattle to Las Vegas. However, Bob brought an extra carry-on item and is going to be charged an additional \$99. In addition, he wants to pick an exit row seat, which costs \$20. He also forgets to print out his tickets beforehand and must do so at the kiosk, costing \$2. Finally, on the flight, he buys some Pringles that cost \$4. What is the percent increase from his original ticket to his final total?
(A) 300 (B) 400 (C) 420 (D) 500 (E) 600

Wesley Wu

Solution: The total cost of his flight is $25 + 99 + 20 + 2 + 4 = 150$ dollars. Thus, the percent increase is $100\left(\frac{150}{25} - 1\right) = \boxed{\text{(D) } 500}$.

5. 2024 congruent squares with side length 2 are drawn in a repeating manner. Below are the leftmost four of the 2024 squares in the figure. Compute the area of the union of all the squares.



(A) 3036 (B) 6072 (C) 6073 (D) 8096 (E) 8097

Wesley Wu

Solution: We can find the answer by counting the 2024 squares and removing the overlap. The area of all 2024 squares is $2024 \cdot 4 = 8096$, and there are 2023 overlapping triangles, each having area 1. Thus, our answer is $8096 - 2023 \cdot 1 = \boxed{\text{(C) } 6073}$.

6. If there exist positive integers $a, b \leq 5$ such that $6ab + 15a + 4b = 62$, find the value of $a + b$.
(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

James Yang

Solution: There really aren't many cases to test. We can also find that a must be even, or else the left-hand side would be odd. Thus, we just have to test $a = 2$ and $a = 4$. The former case yields the solution $(2, 2)$, while the latter case yields no integer solution. Our answer is thus $2 + 2 = \boxed{\text{(A) } 4}$.

7. Wesley, Ryan, Alex, and Patrick are playing FTW for the first time in three months. In FTW, everyone must score a integer amount of points between 0 and 30, inclusive. Wesley scores 16 points in one game of FTW, while the second highest-scoring person has half that amount. How many scores in this game of FTW can there be? For example, one possible score is Wesley: 16, Ryan: 8, Alex: 5, Patrick: 0. Note that ties are allowed.
(A) 24 (B) 45 (C) 147 (D) 168 (E) 217

Wesley Wu

Solution: We know that Wesley's score is fixed and that the second-highest score is 8. We can count the number of scorelines by taking the number of ways

to select three scores between 0 to 8 inclusive and subtract the number of ways to select three scores between 0 and 7 inclusive. This will guarantee that there is a score of 8. Our answer is $9^3 - 8^3 = \boxed{\text{(E) } 217}$.

8. In a 4-digit number, the sum of the first two digits is 5, the sum of the middle two digits is 7, and the sum of the last two digits is 7. Find the sum of all possible values of this number.
(A) 10955 (B) 11190 (C) 14340 (D) 16260 (E) 16785

Ryan Tang

Solution: Let the first digit be a. Then, the second digit is $5 - a$, the third digit is $2 + a$, and the fourth digit is $5 - a$. Note that this number can exist only if $a = 1, 2, 3, 4, 5$. This results in the numbers $1434, 2343, 3252, 4161$, and 5070. Add them up to get $\boxed{\text{(D) } 16260}$.

9. Suppose that X, O, K, S, R, and B are distinct digits. Suppose

$$\overline{XOOKS} + \overline{RBO} = 10434.$$

Compute $X + O + O + K + S + R + B + O$.
(A) 8 (B) 14 (C) 17 (D) 18 (E) 21

Ryan Tang

Solution: Note that X has to be 1 and O has to be 0, since otherwise, the sum would be too big. This results in $\overline{KS} + \overline{RB0} = 434$. Because of unit digits, $S = 4$. Also, $R = 3$, because R cannot be equal to 4, and otherwise, the sum would be too small. As a result, $\overline{K4} + \overline{3B0} = 304 + 10K + 10B = 434$, so $K + B = 13$. As a result, $X+O+O+K+S+R+B+O = 1+0+0+8+4+3+5+0 = \boxed{\text{(E) } 21}$.

10. Positive integers a, b, c, d satisfy $a + b + c + d = 20$ and $a^3 + b^3 + c^3 + d^3 = 530$. Find the value of $a^2 + b^2 + c^2 + d^2$.
(A) 100 (B) 102 (C) 104 (D) 275 (E) 400

Wesley Wu

Solution: The minimum possible value of $a^3+b^3+c^3+d^3$ given $a+b+c+d = 20$ is 500 and occurs at $a = b = c = d = 5$. Notice that this is very close to the 530 we are given. Thus, we try $(4, 5, 5, 6)$, which works. We get $4^2+5^2+5^2+6^2 = \boxed{\text{(B) } 102}$.

11. In the expansion of $(x^8 + x^5 + x^3 + x^2 + x + 1)(x^8 + x^5 + x^3 + x^2 + x - 2024)$, how many terms have a positive coefficient?
(A) 7 (B) 8 (C) 10 (D) 24 (E) 30

Ryan Tang

Solution: Note that the only way for a coefficient to be negative is when the degree is one of $8, 5, 3, 2, 1, 0$, since they would be multiplied by the -2024. Otherwise, all the other coefficients are positive. Through bash, we get that all the other degrees are $16, 13, 11, 10, 9, 7, 6, 4$. This has $\boxed{\text{(B) } 8}$ terms.

12. Call a positive integer an "emordnilap" if it reads the same when it is reflected about a horizontal line. For example, 18 is an emordnilap. How many five digit numbers are either palindromes or emordnilaps?
(A) 270 (B) 1620 (C) 1668 (D) 1860 (E) 2024

Kaiyuan Mao

Solution: Let the set of the five digit palindromes be A and the set of the five digit emordnilaps be B. By PIE the answer is just

$$|A| + |B| - |A \cap B|.$$

Now we calculate each of these values. Constructively counting we find $|A| = 9 \cdot 10^2$.

Now for the emordnilaps; we can consider the possible digits independently. The digits 0, 1, 3, 8 stay the same when they are reflected about the horizontal line, so a number is an emordnilap if and only if it contains only the digits 0, 1, 3, or 8. The first digit can't be 0, so there are four options; for the next four digits they can be any of the five possible digits, so $|B| = 3 \cdot 4^4$.

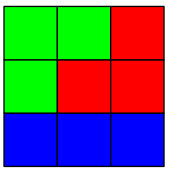
Finally using the same logic we get $|A \cap B| = 3 \cdot 4^2$. Hence, the final answer is $900 + 768 - 48 = \boxed{\text{(B) } 1620}$.

13. Find the sum of the digits of $116^2 - 85^2 + 12^2 + 9^2 + 4^2 + 3^2$.
(A) 7 (B) 10 (C) 15 (D) 16 (E) 19

Ryan Tang

Solution: By Pythagorean triples, we get that $116^2 - 85^2 + 12^2 + 9^2 + 4^2 + 3^2 = 116^2 - 85^2 + 12^2 + 5^2 + 9^2 = 116^2 - 85^2 + 13^2 + 9^2 = 116^2 - 84^2 + 9^2 = (116+84)(116-84)+81 = 6481$, so the sum of the digits is $6+4+8+1 = \boxed{\text{(E) } 19}$.

14. How many ways are there to arrange the numbers 1 to 9 in this 3 by 3 grid so the numbers in the red squares sum to 7 and the numbers in the green squares sum to 22?



(A) 216 (B) 432 (C) 648 (D) 864 (E) 1296

Ryan Tang

Solution: Note that the numbers in the red squares have to be $1, 2, 4$. Also, note that the numbers in the green squares have to be $9, 7, 6$ or $9, 8, 5$. This mames the numbers in the green squares fixed. As a result, this has $3! \cdot (2 \cdot 3!) \cdot 3! =$ $\boxed{\text{(B) } 432}$ ways.

15. In triangle ABC with $\angle A = 90°$, the altitude AD is drawn with D on BC. The angle bisectors of angles BAD and CAD intersect BC at P and Q, respectively. If $AB = 5$ and $AC = 12$, find PQ.
(A) 4 (B) $\frac{60}{13}$ (C) $\frac{63}{13}$ (D) $\frac{65}{13}$ (E) 5

Wesley Wu

Solution: The length of altitude AD is equal to $\frac{5 \cdot 12}{13} = \frac{60}{13}$. Also, $BD = \frac{25}{13}$ and $CD = \frac{144}{13}$ by the Pythagorean theorem or similar triangles. Using the angle bisector theorem, $PD = \frac{25}{13} \cdot \frac{\frac{60}{13}}{\frac{60}{13}+5} = \frac{25}{13} \cdot \frac{12}{25} = \frac{12}{13}$. Similarly, $QD = \frac{144}{13} \cdot \frac{\frac{60}{13}}{\frac{60}{13}+12} = \frac{144}{13} \cdot \frac{5}{18} = \frac{40}{13}$. Adding these together gives $\frac{12}{13} + \frac{40}{13} =$ $\boxed{\text{(A) } 4}$.

16. Call a positive integer n *flippers* if n, when written in base 3, contains an equal amount of 0's, 1's, and 2's. How many *flippers* numbers are there from 1 to 3^9, inclusive?
(A) 1184 (B) 1440 (C) 1620 (D) 1776 (E) 1945

Wesley Wu

Solution: There are three cases here–the 9-digit *flippers* numbers, the 6-digit ones, and the 3-digit ones (in base three). Notice that the 9-digit case is simply the number of ways to rearrange the numbers in the string 000111222. This is counted by $\frac{9!}{3!3!3!}$. Similarly, the other cases are $\frac{6!}{2!2!2!}$ and $\frac{3!}{1!1!1!}$. This is a total of 1776.
However, these cases all count when the leading digit (in ternary) equals 0, which would remove a zero. For example, the number 012 would simply be written as 12, which doesn't satisfy the conditions. Each number (0, 1, or 2) has an equal chance of being the first number, so we can simply multiply by $\frac{2}{3}$ to get $\boxed{\text{(A) } 1184}$.

17. For how many ordered pairs of positive integers (a, b) with $a, b \leq 100$ is the condition $|a - b| < \frac{|a+b|}{2}$ satisfied?
(A) 3300 (B) 3400 (C) 4500 (D) 6600 (E) 6700

Wesley Wu

Solution: First, note that $a+b$ is always positive, so the absolute value signs are redundant. Next, we split this into three cases to deal with the left-hand side.
Suppose $a = b$. Then, the inequality becomes $0 < \frac{a+b}{2}$, which is satisfied by all pairs of integers (k, k) in our given range. This adds 100 cases.
Next, let's consider $a > b$ or $a < b$, which are symmetric. Without loss of generality, let $a > b$. Then, $2a-2b < a+b$ and $a < 3b$. We can combine these inequalities to find $b < a < 3b$. Let's casework on b. If b is between 1 and 33 inclusive, all numbers a in that range work, for a total of $3b-b-1 = 2b-1$ possibilities (since the bounds are strict). This gives a total of $2 \cdot \frac{33 \cdot 34}{2} - 33 = 33 \cdot 33$. If $b > 33$, the upper bound becomes 100 instead. Thus, we have $100 - b$ cases (because one bound is strict and the other is not). This is an arithmetic sequence from 66 to 1. Thus, its sum is $\frac{66 \cdot 67}{2} = 33 \cdot 67$. Combining the two cases gives $33 \cdot 100 = 3300$ cases.
Finally, we multiply this by 2 to restore generality, and we add back the $a = b$ to get $\boxed{\text{(E) } 6700}$.

18. Call a positive integer "orz" if it equals the sum of its digits cubed. For example, 5832 is orz because $5832 = (5 + 8 + 3 + 2)^3$. What is the sum of the first three orz integers?
(A) 1242 (B) 2710 (C) 5426 (D) 7622 (E) 11257

Ryan Tang

Solution: Let n be the sum of the digits. Take mod 9 to get that $n \equiv n^3 \pmod 9$, since n is equivalent to the sum of its digits mod 9. As a reuslt, $n^3 - n \equiv n(n+1)(n-1) \equiv 0 \pmod 9$. Testing $n = 1, 8, 9, 10, 17$ gives that $n = 1, 8, 17$ work, which cause the numbers to be $1, 512, 4913$, and the answer is $1 + 512 + 4913 = \boxed{\text{(C) } 5426}$.

19. Ryan, Wesley, and Patrick are each uniformly randomly throwing one dart on a continuous number line from -10 to 10. Their score is defined as the distance from the number they hit to 3. What is the probability that the sum of their scores is less than 5?
(A) $\frac{4}{375}$ (B) $\frac{43}{3087}$ (C) $\frac{1}{48}$ (D) $\frac{11}{441}$ (E) $\frac{1}{24}$

Ryan Tang

Solution: Let a, b, c be the hit numbers. We know that $|a-3|+|b-3|+|c-3| < 5$. Note that this is an octahedron with side length 5. The area of one of the sub-tetrahedra is equal to $\frac{5^3}{6}$, and there are 8 of them, so the total area is $\frac{500}{3}$, and the probability is $\frac{\frac{500}{3}}{20^3} = \boxed{\text{(C) } \frac{1}{48}}$.

20. Quadrilateral $ABCD$ has $AB = AD$, $CB = CD$, and $\angle B = \angle D = 90^\circ$. It is also given that $ABCD$ has an incenter and a circumcenter. The incenter of this quadrilateral is denoted I, and the circumcenter of this quadrilateral is denoted O. If $IO = 425$ and $AC = 1550$, compute the inradius of $ABCD$.
(A) 310 (B) 336 (C) 425 (D) 480 (E) 552

Wesley Wu

Solution: $\boxed{\text{(B) } 336}$.

21. Let a, b, c be the roots of $f(x) = x^3 - 4x^2 + 5x - 8$. Find $f(ab)f(bc)f(ac)$.
(A) -71424 (B) -46080 (C) -21504 (D) 10 (E) 81

Ryan Tang

Solution: Since $f(x)$ is monic, we can factor it into $(x-a)(x-b)(x-c)$. In addition, since $abc = 8$, $ab = \frac{8}{c}$, $bc = \frac{8}{a}$, and $ac = \frac{8}{b}$. Therefore, we want to calculate the product

$$\left(\frac{8}{a} - a\right)\left(\frac{8}{b} - b\right)\left(\frac{8}{c} - c\right)\left(\frac{8}{a} - b\right)\left(\frac{8}{b} - c\right)\left(\frac{8}{c} - a\right)\left(\frac{8}{b} - a\right)\left(\frac{8}{c} - b\right)\left(\frac{8}{a} - c\right).$$

Let's split this into groups of 3. First, we'll calculate $\left(\frac{8}{a} - b\right)\left(\frac{8}{b} - c\right)\left(\frac{8}{c} - a\right)$ and $\left(\frac{8}{b} - a\right)\left(\frac{8}{c} - b\right)\left(\frac{8}{a} - c\right)$. Expanding, we find that both are equal to $\frac{512}{abc} - 64\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + 8(a+b+c) - abc = 64 - 64 \cdot \frac{5}{8} + 8 \cdot 4 - 8 = 48$.
Now, let's calculate $\left(\frac{8}{a} - a\right)\left(\frac{8}{b} - b\right)\left(\frac{8}{c} - c\right)$. This is equal to

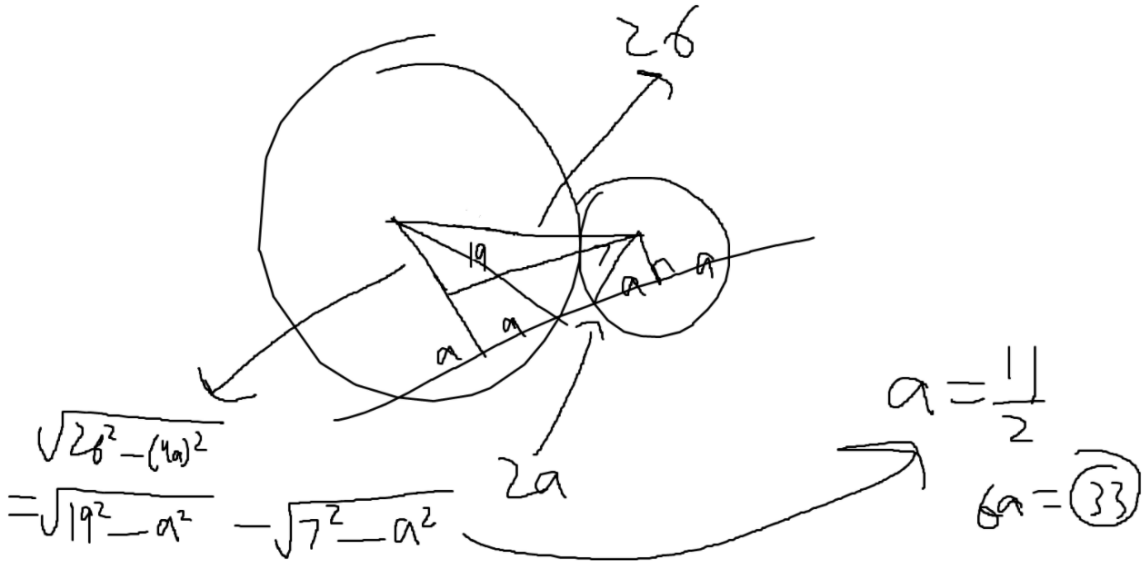
$$\frac{(8-a^2)(8-b^2)(8-c^2)}{abc} = \frac{512 - 64(a^2+b^2+c^2) + 8(a^2b^2+b^2c^2+c^2a^2) - (abc)^2}{abc}$$

$$= \frac{512 - 64((a+b+c)^2 - 2(ab+bc+ca)) + 8((ab+bc+ca)^2 - 2abc(a+b+c)) - (abc)^2}{abc}$$

$$= \frac{512 - 64(6) + 8(-39) - 64}{8} = 64 - 48 - 39 - 8 = -31.$$

Finally, our answer is $48 \cdot 48 \cdot (-31) = \boxed{\text{(A) } -71424}$.

22. Two circles with radius 19 and 7, respectively, are externally tangent to each other. A line ℓ passing through both circles intersects the larger circle at A and B and



the smaller circle at C and D with B on AC and C on BD. If $AB = BC = CD$, compute the length of AD.
(A) 11 (B) 18 (C) 24 (D) 27 (E) 33

Wesley Wu

Solution:

23. In Yaptrick's triangle, it starts with a 1, and then the next term is one more than the sum of the two terms above it. The numbers on the edge are always 1. The first few rows look like this:

1	(Row 0)
1 1	(Row 1)
1 3 1	(Row 2)
1 5 5 1	(Row 3)

How many times does the number 239 appear in Yaptrick's triangle?
(A) 2 (B) 3 (C) 4 (D) 6 (E) 8

Ryan Tang

Solution: Let's focus on the diagonals instead. Let the 0th diagonal be $1, 1, 1...$, let the 1st diagonal be $1, 3, 5, ...$, and so on. We claim that the nth item of the

mth diagonal equals to $2\binom{n}{m-1}$. This can be easily proven by induction.
Our base case is the 1st item of the 0th diagonal, which is is $2\binom{1}{0} - 1 = 1$.
Then, for our inductive step, the $n+1$th item equals $(2\binom{n}{m}-1)+(2\binom{n}{m-1}-1)+1 = 2\binom{n+1}{m} - 1$ by Pascal's identity.

For an item to equal 239, $2\binom{n}{m} - 1 = 239$, so $\binom{n}{m} = 120$. We can just consider $n \geq 2m$ by symmetry, and multiply by 2 at the end, except for the cases where $n = 2m$.
Note that for $m = 1$, $(120, 1)$ is an obvious and unique solution. For $m = 2$, we get that $\frac{n(n-1)}{2} = 120$. Solving gives $n^2 - n - 240 = 0$, so $(n - 16)(n + 15) = 0$ and $(16, 2)$ works.
For $m = 3$, note that $\binom{10}{3} = 120$ using bash starting with $n = 6$, so $(10, 3)$ works.
For $m = 4$, we bash values of n starting with 8 to get that 8 doesn't work and $\binom{9}{4} > 120$, so there are no solutions.
For $m = 5$, note that $\binom{10}{5} > 120$, so there are no solutions.
As a result, there are 3 solutions, none of which satisfy $n = 2m$, so we multiply by 2 due to symmetry to get our final answer of $\boxed{\text{(D) } 6}$ solutions.

24. In isosceles trapezoid $ABCD$ with AB and CD parallel which has an incircle, the diagonal BD intersects the incircle of $ABCD$ at points P and Q, with P closer to D than Q is. Given that $DP = 3$ and $QB = 1$, the length of PQ can be expressed as $\sqrt{a + b\sqrt{c}} - d$, where a, b, c, and d are positive integers, and c is square-free. Compute $a + b + c + d$.
(A) 25 (B) 92 (C) 176 (D) 224 (E) 358

25. For how many ordered triples (a, b, c) of positive integers $1 \leq a, b, c \leq 30$ is $\frac{(a+b+c)^2}{a^2+b^2+c^2}$ noninteger?
(A) 26913 (B) 26925 (C) 26946 (D) 26970 (E) 27000

Wesley Wu

Solution: First, let's count the number of ordered triples such that $\frac{(a+b+c)^2}{a^2+b^2+c^2}$ is an integer. Note that by Cauchy-Schwarz, $(a^2 + b^2 + c^2)(1 + 1 + 1) \geq (a + b + c)^2$. Thus, $\frac{(a+b+c)^2}{a^2+b^2+c^2} \leq 3$. Also, $\frac{(a+b+c)^2}{a^2+b^2+c^2} > 1$. Thus, $\frac{(a+b+c)^2}{a^2+b^2+c^2} = 2, 3$. We shall tackle these cases separately.
Suppose $\frac{(a+b+c)^2}{a^2+b^2+c^2} = 3$. Then, equality holds in the Cauchy inequality, and thus $a = b = c$. This provides 30 solutions.
Now, suppose $\frac{(a+b+c)^2}{a^2+b^2+c^2} = 2$. This means that $a^2 + b^2 + c^2 - 2ab - 2ac - 2bc = 0$, or $c^2 - c(2a + 2b) + (a^2 + b^2 - 2ab) = 0$. Using the quadratic formula, we find $c = \frac{2a+2b\pm\sqrt{4a^2+4b^2-4a^2-4b^2+8ab}}{2} = \frac{2a+2b\pm2\sqrt{2ab}}{2} = a + b \pm \sqrt{2ab}$. Thus, $c = (\sqrt{a} + \sqrt{b})^2$ or $(\sqrt{a} - \sqrt{b})^2$. This provides the cases $\sqrt{c} = \sqrt{a} + \sqrt{b}$ and permutations. Let's casework again, this time on the number of permutations there are

of (a, b, c).

Suppose that there are 3 permutations. Then, two of the three must be equal. Clearly, the smaller two must be equal, so our equation becomes $\sqrt{c} = 2\sqrt{a}$, so our solutions for this case are $(n, n, 4n)$, which provides $3 \cdot 7 = 21$ cases.

Finally, let a, b, c be distinct. Then, note that all solutions are of the form $(kx^2, ky^2, k(x+y)^2)$. We can first find the perfect square solutions, which are $(1, 4, 9), (1, 9, 16), (1, 16, 25), (4, 9, 25)$. Including the constant, our other two solutions are $(2, 8, 18), (3, 12, 27)$. This adds $6 \cdot 6 = 36$ solutions.

There are 27000 total possible triples, and of them, $30 + 21 + 36 = 87$ triples are integers. Thus, there are $27000 - 87 = \boxed{\text{(A) } 26913}$ total triples where the fraction is noninteger.